

# WHERE TO PLACE A HOLE TO ACHIEVE A MAXIMAL ESCAPE RATE.

LEONID A. BUNIMOVICH AND ALEX YURCHENKO

**ABSTRACT.** A natural question of how the survival probability depends upon a position of a hole was seemingly never addressed in the theory of open dynamical systems. We found that this dependency could be very essential. The main results are related to the holes with equal sizes (measure) in the phase space of strongly chaotic maps. Take in each hole a periodic point of minimal period. Then the faster escape occurs through the hole where this minimal period assumes its maximal value. The results are valid for all finite times (starting with the minimal period) which is unusual in dynamical systems theory where typically statements are asymptotic when time tends to infinity. It seems obvious that the bigger the hole is the faster is the escape through that hole. Our results demonstrate that generally it is not true, and that specific features of the dynamics may play a role comparable to the size of the hole.

## 1. INTRODUCTION.

The theory of open dynamical systems is (naturally) much less developed than of the closed ones. Basically so far the problems studied were on the existence of conditionally invariant measures, their properties, and the existence of the escape rates [5], [7], [10], [11], [12], [14], [15], [28], [33], [35].

In this paper we address a natural question which, to the best of our knowledge, has not been studied so far. Obviously, if one enlarges a hole then the escape rate of the orbits will increase as well (or, at least, it cannot decrease). Consider, however, two holes of the same size (measure), placed at the different positions in the phase space of the dynamical system under study. Would the escape rates through these holes be equal?

We demonstrate that the answer to this question could be both "yes" and "no". In case when there exists a group of measure preserving translations of the phase space which commute with the dynamics the answer is "yes". It is quite natural and intuitive answer which is justified in Section 3 of our paper. However the dynamics of these systems is quite regular.

Much less trivial is the question of what other factors, besides the size of the hole, can influence the escape rate. In particular, what can generate different escape rates through two holes of the same size?

Consider a system with strongly chaotic dynamics. For many classes of such systems it is known that there exists infinitely many periodic orbits of infinitely many periods and that the periodic orbits are everywhere dense in the phase space. Therefore in each hole there are infinitely many periodic points.

Our approach is based on the idea that the faster escape occurs through a hole whose preimages overlap less than the ones of another hole. This idea leads to the following procedure (algorithm): 1) find in each hole a periodic point with minimal period; 2) compare these periods. We claim that the escape will be faster through a hole where this minimal period is bigger. This claim is justified for various classes of dynamical systems with strongly chaotic behavior and Markov holes in Section 4.

We also computed the local escape rate and demonstrated that for all nonperiodic points this value is the same while at the periodic points the escape "slows down" and it assumes smaller values at the periodic points with smaller period.

Thus we demonstrate that the dynamical factors could be as important for the escape as the size of the hole. In fact it is possible that the escape rate through a larger hole could be less than the escape rate through a smaller hole.

An important new feature of our results is that they hold for all finite times (starting with some moment) in comparison to the usual setup in the theory of dynamical systems where one deals with the asymptotic properties at infinite time.

For more general classes of dynamical systems not only the distribution of the periodic points, but other characteristics of dynamics, e.g. distortion, may also contribute to the process of escape. This will be considered in a future publication.

The structure of the paper is the following one. Section 2 contains necessary definitions and some auxiliary results. Section 3 deals with the case where escape rate does not depend on the position of the hole. Section 4 presents the main results of the paper. Section 5 deals with some generalization and, finally, Section 6 contains concluding remarks.

## 2. DEFINITIONS AND SOME TECHNICAL RESULTS.

Consider a discrete time dynamical system given by a measure-preserving map  $\hat{T} : \hat{M} \rightarrow \hat{M}$ , where  $\hat{M}$  is a Borel probability space with the measure  $\lambda$ . Let  $\mathcal{B}$  be the Borel  $\sigma$ -algebra on  $\hat{M}$  with respect to  $\lambda$ .

**2.1. Recurrences.** Here we define some notions related to the recurrence properties of the dynamical system.

**Definition 2.1.1.** *The Poincaré recurrence time of a subset  $A \in \mathcal{B}$  of a positive measure is a positive integer  $\tau(A) \leq +\infty$  given by*

$$(2.1) \quad \tau_{\hat{T}}(A) = \min_{n \geq 1} \{n : \lambda(\hat{T}^{-n}(A) \cap A) > 0\}.$$

If there is no ambiguity about which map we are considering, then we drop the subscript and use  $\tau(A)$  instead. There is a related topological notion, also called Poincaré recurrence time, defined as (see [2])  $\tilde{\tau}_{\hat{T}}(A) = \inf_{n \geq 1} \{n : \hat{T}^n(A) \cap A \neq \emptyset\}$ .

Next, we list a few properties of Poincaré recurrence time which will be used later. These statements follow easily from the definition and they are similar to the properties of the topological Poincaré recurrence time (see Lemma 2 in [2]).

**Proposition 2.1.2.** *Let  $A$  and  $B$  be two measurable sets. Then*

- a) *if  $A \subset B$  then  $\tau(A) \geq \tau(B)$ ;*
- b) *if  $\hat{T}$  maps sets of measure zero to sets of measure zero, then  $\tau(A) = \tau(\hat{T}^{-1}(A))$ , where  $\hat{T}^{-1}(A)$  is a complete preimage of  $A$ .*

For  $n \geq 0$  and  $A \in \mathcal{B}$ , define the following (measurable) sets [15],

$$\begin{aligned}\Omega_n(A) &= \left\{x \in \hat{M} : \exists j \in \mathbb{N}, 0 \leq j \leq n, \hat{T}^j x \in A\right\} = \cup_{i=0}^n \hat{T}^{-i}(A), \\ \Theta_n(A) &= \left\{x \in \hat{M} : \hat{T}^n x \in A, \hat{T}^j x \notin A, j = 0, \dots, n-1\right\},\end{aligned}$$

where  $\hat{T}^{-i}(A)$  is a complete preimage of  $A$  under  $\hat{T}^i$ . The set  $\Omega_n(A)$  consists of all points which orbits enter  $A$  after no more than  $n$  iterates. The set  $\Theta_n(A)$  consists of all points which orbits enter  $A$  at first time exactly on  $n$ th iterate. Note that  $\Omega_0(A) = A$  and  $A \subset \Omega_n(A)$ ,  $\forall n \in \mathbb{N}$ . It is easy to see that these sets have the following properties.

**Proposition 2.1.3.** *Let  $A$  be a measurable set. Then*

- a)  $\Omega_i(A)$  is a nondecreasing sequence of sets;
- b)  $\Theta_i(A) \cap \Theta_j(A) = \emptyset$  if  $i \neq j$ ,  $i, j > 0$ ;
- c)  $\Omega_n(A) = \cup_{i=0}^n \Theta_i(A)$ .

**2.2. Open dynamical systems.** Let  $A$  be a measurable set and let  $M = \hat{M} \setminus A$ . We define an open dynamical system (system with a "hole"  $A$ ) as a map  $T : M \rightarrow \hat{M}$ , where  $T := \hat{T}|_M$  is a restriction of  $\hat{T}$  to  $M$ . We keep track of the orbits while they stay outside the "hole"  $A$ , and after they enter a hole we no longer consider these orbits (they just "disappear"). So we can talk about iterates of  $T$  instead of  $\hat{T}$  as long as orbit stays outside  $A$ . Alternatively, one can redefine  $\hat{T}$  in such a way that it is an identity map on  $A$ . We will use the former approach. Note that we use a hat over a letter to denote an object in a closed system and letters without a hat for corresponding objects in the open system.

### 2.3. Escape rate.

**Definition 2.3.1.** *The (exponential) escape rate into the hole  $A$  is a nonnegative number  $\rho(A)$  given by*

$$(2.2) \quad \rho(A) = - \lim_{n \rightarrow \infty} \frac{1}{n} \ln \lambda \left( \hat{M} \setminus \Omega_n(A) \right),$$

if this limit exists.

The number  $\lambda \left( \hat{M} \setminus \Omega_n(A) \right) = 1 - \lambda(\Omega_n(A))$  (sometimes called a *survival probability*) is the measure of the set that does not escape into the hole in  $n$  iterations. Hence, the escape rate represents the average rate at which orbits enter the hole. The larger the escape rate is, the faster the "mass" escapes from the system into the hole  $A$ .

We will only consider systems in which almost every orbit eventually enters the hole, i.e. systems which satisfy the following condition

$$(H1) \quad \sum_{i=0}^{\infty} \lambda(\Theta_i(A)) = 1.$$

Any ergodic system would be an example of such a system. In that case property H1 holds for any measurable hole of positive measure. On the other hand if we consider a system with a globally attracting set  $A$  then the property H1 holds only for that set  $A$  and any set which contains  $A$ .

The next proposition lists a few simple but useful properties of the escape rate.

**Proposition 2.3.2.** *Let  $A$  and  $B$  be two measurable sets. Assume that  $\rho(A)$  and  $\rho(B)$  exist. Then,*

a)

$$\rho(A) = - \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left( 1 - \sum_{i=0}^n \lambda(\Theta_i(A)) \right);$$

b) if  $A \subset B$  then  $\rho(A) \leq \rho(B)$ ;

c)  $\rho(A) = \rho(\hat{T}^{-1}(A))$ ;

d) for any finite positive integer  $m$ ,  $\rho(A) = \rho\left(\bigcup_{i=0}^m \hat{T}^{-i} A\right)$ ;

e) if  $B \subset \hat{T}^{-k} A$  for some  $k > 0$  then  $\rho(A \cup B) = \rho(A)$ ;

f) if there exists  $m$  such that  $\hat{M} = \bigcup_{i=0}^m \hat{T}^{-i} A$  then  $\rho(A) = +\infty$ .

The first part of the proposition says that the size of the hole is one of the factors that determines the escape rate. As we will see later, it is not necessarily the only one or even the dominant one. Moreover, d) and e) state that, in principle, we can have a system in which holes of different size have the same escape rate.

Recall now the notion of metric conjugacy which will play an important role in what follows. Note that for Lebesgue probability spaces metric conjugacy is equivalent to two maps being isomorphic (see, for example, Theorem 2.5 and 2.6 in [36]). We use the following definition.

**Definition 2.3.3.** *Let  $T_i$  be a measure-preserving transformation of the Lebesgue probability space  $(X_i, \mathcal{B}_i, \lambda_i)$ ,  $i = 1, 2$ , where  $\mathcal{B}_i$  is a Borel  $\sigma$ -algebra on  $X_i$  and  $\lambda_i$  is a probability measure. We say that  $T_1$  and  $T_2$  are metrically conjugate if there exist  $M_i \in \mathcal{B}_i$  with  $\lambda_i(M_i) = 1$  and  $T_i(M_i) \subset M_i$  and there is a invertible measure-preserving transformation (called metric conjugacy)  $F : M_2 \rightarrow M_1$  such that*

$$F \circ T_2(x) = T_1 \circ F(x), \quad \forall x \in M_2.$$

The following simple statement says that if two systems are metrically conjugate, then the escape rates into the corresponding holes and Poincaré return times of these holes are the same for both systems.

**Lemma 2.3.4.** *Let  $T_1$  and  $T_2$  be two metrically conjugate measure-preserving transformations on the Borel probability spaces  $(X_1, \mathcal{B}_1, \lambda_1)$  and  $(X_2, \mathcal{B}_2, \lambda_2)$ , correspondingly, with a conjugacy map  $F : (\mathcal{B}_2, \lambda_2) \rightarrow (\mathcal{B}_1, \lambda_1)$ . Suppose also that  $T_2$  satisfy condition H1. Then  $\forall A \in \mathcal{B}_2$  we have*

a)  $\rho_{T_2}(A) = \rho_{T_1}(F(A))$ ,

b)  $\tau_{T_2}(A) = \tau_{T_1}(F(A))$ .

*Proof.* a) Let  $A \in \mathcal{B}_2$  and, as above, define two sets

$$\begin{aligned} \Theta_n^2(A) &= \left\{ x \in X_2 : T_2^n x \in A, T_2^j x \notin A, j = 0, \dots, n-1 \right\}; \\ \Theta_n^1(F(A)) &= \left\{ y \in X_1 : T_1^n y \in F(A), T_1^j y \notin F(A), j = 0, \dots, n-1 \right\}. \end{aligned}$$

It is easy to see that  $F(\Theta_n^2(A)) = \Theta_n^1(F(A))$ ,  $\forall n \geq 1$ . Therefore,  $\lambda_2(\Theta_n^2(A)) = \lambda_1(F(\Theta_n^2(A))) = \lambda_1(\Theta_n^1(F(A)))$ . Hence, one has that  $\rho_{T_2}(A) = \rho_{T_1}(F(A))$ ,  $\forall A \in \mathcal{B}_2$ .

- b) Suppose that  $\tau_{T_2}(A) = N < \infty$ . That means that  $\lambda(T_2^N(A) \cap A) > 0$  and  $\lambda(T_2^k(A) \cap A) = 0$ ,  $k < N$ . But

$$\lambda(T_1^N(F(A)) \cap F(A)) = \lambda(F(T_2^N(A)) \cap F(A)) > 0$$

and similarly  $\lambda(T_1^k(F(A)) \cap F(A)) = 0$ ,  $k < N$ . Hence,  $\tau_{T_1}(F(A)) = N$ .  $\square$

### 3. ESCAPE RATE FOR THE ERGODIC GROUP ROTATIONS.

Suppose that the phase space  $\hat{M}$  is a compact connected metric group. Let  $S_a : \hat{M} \rightarrow \hat{M}$  be a group rotation defined as  $S_a(x) = ax$ , for some  $a \in \hat{M}$ . Then there is the unique any rotation invariant probability measure,  $\lambda$ , called Haar measure (see, e.g. [25]).

The following simple statement claims that if a group rotation  $S$  commutes with the dynamics, i.e.

$$(H2) \quad \hat{T}^{-1}S = S\hat{T}^{-1},$$

then the escape rate is invariant when we rotate the hole by  $S$ .

**Lemma 3.0.5.** *Suppose that  $\hat{T}^{-1}S(A) = S\hat{T}^{-1}(A)$  for some  $S \in \mathbb{G}$  and  $A \in \mathbb{B}$ . Also, assume that  $\rho_{\hat{T}}(A)$  exists. Then  $\rho_{\hat{T}}(A) = \rho_{\hat{T}} \circ S(A)$ .*

*Proof.* Let  $A \in \mathbb{B}$  as in the statement of the theorem. Then,

$$\begin{aligned} \Theta_n(S(A)) &= \left\{ x \in \hat{M} : \hat{T}^n x \in S(A), \hat{T}^j x \notin S(A), j = 0, \dots, n-1 \right\} \\ &= \left\{ x \in \hat{M} : \hat{T}^n S^{-1}x \in A, \hat{T}^j S^{-1}x \notin A, j = 0, \dots, n-1 \right\} \\ &= \left\{ Sy \in \hat{M} : \hat{T}^n y \in A, \hat{T}^j y \notin A, j = 0, \dots, n-1 \right\} = S(\Theta_n(A)). \end{aligned}$$

$\square$

Assume now that  $\hat{T} = \hat{T}_a$  is an ergodic rotation of  $\hat{M}$  given by  $\hat{T}_a(x) = ax$ , for some  $a \in \hat{M}$ . For the rotations we use the following property as a definition of ergodicity.

**Proposition 3.0.6** (see, for example, Theorem 1.9 in [36]). *Let  $\hat{M}$  be a compact group and  $\hat{T}_a$  a rotation of  $\hat{M}$ . Then  $\hat{T}_a$  is ergodic iff  $\{a^n\}_{n=-\infty}^{+\infty}$  is dense in  $\hat{M}$ .*

The simplest example of this class of dynamical systems is the irrational rotations of the circle. It is known (see, for example, Theorem 1.9 in [36]) that if there is an ergodic rotation of  $\hat{M}$  then  $\hat{M}$  must be Abelian. In that case conditions H1 and H2 are satisfied so Lemma 3.0.5 is applicable. Hence, the escape rate for any hole is independent of the position of that hole. Moreover, one can compute the escape rate for any hole that contains an open set. It turns out that it is infinite because in the case of ergodic rotation all orbits escape within finite amount of time.

**Theorem 3.0.7.** *For any ergodic rotation  $\hat{T}_a = ax$  of the compact connected metric group  $\hat{M}$  and any hole  $A$  that contains an open ball the escape rate is infinite.*

*Proof.* Let  $\text{dist}(\cdot, \cdot)$  be a metric on  $\hat{M}$ . Suppose that  $V(x, \varepsilon)$  is an open ball of radius  $\varepsilon$  centered at  $x$  that is contained in the hole  $A$ . Since  $\hat{M}$  is a compact metric space we can find a finite covering by open balls of radius  $\varepsilon/4$ . Let  $\{V_i(x_i, \varepsilon/4)\}_{i=1}^m$  be that covering.

It follows from ergodicity that if  $\hat{T}_a$  is ergodic then the set  $\{a^i\}_{i=1}^{+\infty}$  is dense in  $\hat{M}$ . Therefore we can find  $n$  and  $\{n_j\}_{j=1}^m$  such that

$$\text{dist}(a^n, x) < \frac{1}{4}\varepsilon; \quad \text{dist}(a^{n_j}, x_j) < \frac{1}{4}\varepsilon, \quad \forall j = 1 \dots m, \quad n > \max_{j=1 \dots m} \{n_j\}.$$

Then we have that

$$\begin{aligned} \text{dist}(\hat{T}_a^{n-n_j} x_j, x) &\leq \text{dist}(\hat{T}_a^{n-n_j} x_j, a^n) + \text{dist}(a^n, x) \\ &= \text{dist}(\hat{T}_a^{n-n_j} x_j, \hat{T}_a^{n-n_j} a^{n_j}) + \text{dist}(a^n, x) = \text{dist}(x_j, a^{n_j}) + \text{dist}(a^n, x) < \frac{1}{2}\varepsilon. \end{aligned}$$

Therefore,  $\forall j = 1 \dots m$ ,  $\hat{T}_a^{n-n_j} V_j \subset V$ . Thus every ball  $V_j$  will be mapped into the hole in finite number of steps. Since every set can be covered by these balls we get that any set is mapped into the hole in finite number of iterations.  $\square$

#### 4. ESCAPE RATE FOR THE EXPANDING MAPS OF THE INTERVAL.

In this section we look at the examples of the dynamical systems in which the position of the hole plays an important role in determining the escape rate. We consider some classes of the uniformly expanding maps of the interval that have a finite Markov partition. These systems are the examples of so called chaotic dynamical systems.

Consider first  $\hat{T} : [0, 1] \rightarrow [0, 1]$  given by

$$\hat{T}x = \kappa x \mod 1,$$

where  $\kappa$  is an integer larger than one. This map preserves the Lebesgue measure on  $[0, 1]$ . Without any loss of generality one can assume that  $\kappa = 2$ .

Fix  $N \in \mathbb{N}$  and let  $\mathcal{I}_N = \{I_{i,N}\}_{i=1}^{2^N}$  be the partition consisting of the pre-images of the elements of Markov partition  $\{[0, 0.5], [0.5, 1]\}$  of  $[0, 1]$  given by

$$I_{i,N} = \left[ \frac{i-1}{2^N}, \frac{i}{2^N} \right], \quad i = 1 \dots 2^N.$$

Define a partition  $\mathcal{I}_N^k = \{I_{j,N+k}\}_{j=1}^{2^{N+k}}$  as  $k^{th}$  preimage of the partition  $\mathcal{I}_N$ , i.e. for each  $j$ ,  $j = 1 \dots 2^{N+k}$ , there is  $i$ ,  $i = 1 \dots 2^N$ , such that  $\hat{T}^k I_{j,N+k} = I_{i,N}$ . Observe that  $\mathcal{I}_N^k$  are Markov partitions themselves.

Consider the open dynamical system defined by the map  $\hat{T}$  and the hole  $I_{i,N}$ ,

$$T_{i,N} : [0, 1] \setminus I_{i,N} \rightarrow [0, 1]$$

as in section 2.2. For each  $i$ ,  $i = 1 \dots 2^N$ , we have different open dynamical system with a corresponding hole  $I_{i,N}$  (we refer to this hole as to a *Markov hole* because  $I_{i,N}$  is an element of Markov partition). Define the Poincaré recurrence time of the hole,  $\tau(I_{i,N})$ , escape rate into the hole,  $\rho(I_{i,N})$ , and the set

$$\Omega_n(I_{i,N}) = \left\{ x \in [0, 1] : \exists j, 0 \leq j \leq n, \quad \hat{T}^j x \in I_{i,N} \right\}, \quad n \geq 0.$$

In Section 4.3 we will show that escape rate is well defined and then, in Section 4.5, that it depends not only on the size of the hole but also on its position. More precisely, we prove the following.

**Theorem 4.0.8** (Main Theorem). *Let  $I_{i,N}$  and  $I_{j,N}$  be two Markov holes for the doubling map. Suppose that  $\tau(I_{j,N}) > \tau(I_{i,N})$ . Then,*

$$\rho(I_{j,N}) > \rho(I_{i,N}).$$

Moreover, for all  $n \geq \tau(I_{i,N})$ ,

$$1 - \lambda(\Omega_n(I_{j,N})) < 1 - \lambda(\Omega_n(I_{i,N})).$$

In the section 4.6 we show that asymptotically as we decrease the size of the hole escape rate is proportional to the size of the hole.

**Theorem 4.0.9** (Local Escape Rate). *Let  $x \in [0, 1]$  and  $\{A_n(x)\}_{n=1}^\infty$  is a sequence of nested decreasing intervals with  $x = \cap_{n=1}^\infty A_n(x)$ . The following statements hold:*

a) *if  $x$  is a periodic point of period  $m$  then*

$$\lim_{n \rightarrow \infty} \frac{\rho(A_n(x))}{\lambda(A_n(x))} = 1 - \frac{1}{2^m};$$

b) *if  $x$  is a non-periodic point and  $x \neq s2^{-k}$ ,  $s, k \in \mathbb{Z}^+$ , then*

$$\lim_{n \rightarrow \infty} \frac{\rho(A_n(x))}{\lambda(A_n(x))} = 1.$$

Moreover, for a sequence of shrinking Markov holes (Section 4.7) the corresponding sequence of escape rates is a monotone one.

**Theorem 4.0.10** (Monotonicity of the escape rate).

$$\max_{1 \leq i \leq 2^{N+1}} \rho(I_{i,N+1}) = \min_{1 \leq j \leq 2^N} \rho(I_{j,N}).$$

First, we state some known results about doubling map and then reformulate the problem in terms of the symbolic dynamics.

**4.1. Preliminary results.** Let  $p_{i,N}(n)$  be a number of points of period (not necessary the minimal one)  $n$  in the hole  $I_{i,N}$ . Due to the fact that  $\mathcal{I}_N$  is a Markov partition, for each  $j$ ,  $j = 1 \dots 2^{N+k}$ ,  $\hat{T}^k(I_{j,N+k}) = I_{s,N}$  for some  $s$ ,  $s = 1 \dots 2^N$ . Let  $f_{i,N}^k$  be a number of elements of partition  $\mathcal{I}^{N+k}$  that do not enter the hole  $I_{i,N}$  in the first  $k$  iterations, i.e.

$$f_{i,N}^k = \# \left\{ j : I_{j,N+k} \in \mathcal{I}_N^k; \lambda \left( \hat{T}^s(I_{j,N+k}) \cap I_{i,N} \right) = 0; s = 0, \dots, k \right\}.$$

The next two results show how to compute the Poincaré recurrence time and the escape rate for any Markov hole.

**Proposition 4.1.1.** *The Poincaré recurrence time of the hole  $I_{i,N}$  is equal to the period of a periodic point contained in  $I_{i,N}$  having the smallest period, i.e.*

$$\tau(I_{i,N}) = \min_{n \geq 1} \{n : p_{i,N}(n) > 0\}.$$

*Proof.* Suppose that a point  $x \neq 0$  is a periodic point of the smallest period in the hole  $I_{i,N}$ . Let that period is equal to  $p > 1$  (the case  $p = 1$  is considered separately). All periodic points have the form  $\frac{l}{2^k - 1}$ ,  $k \in \mathbb{N}$ ,  $l \in \mathbb{N}$ ,  $0 \leq l \leq 2^k - 1$ . The endpoints of the elements of a Markov partition have the form  $\frac{n}{2^s}$ ,  $k \in \mathbb{N}$ ,  $n \in \mathbb{N}$ ,  $0 \leq n \leq 2^k - 1$ . Therefore all periodic points except for zero and one are in the interior of the elements of the partition. Thus  $\lambda(\hat{T}^p(I_{i,N}) \cap I_{i,N}) > 0$ , and, therefore,  $\tau(I_{i,N}) \leq p$ .

To obtain the opposite inequality we need to use the Markov property of the partition. Suppose that  $\lambda(\hat{T}^k(I_{i,N}) \cap I_{i,N}) > 0$  for some  $k \leq p$ . Since partition is Markov we have  $I_{i,N} \subset \hat{T}^k(I_{i,N})$ . Thus by Intermediate Value Theorem we conclude that  $I_{i,N}$  contains a periodic point of period  $k$ . Therefore,  $\tau(I_{i,N}) \geq p$ .

Now consider the case of a fixed point,  $x = 0$  ( $x = 1$  can be treated similarly). Clearly,  $\lambda(\hat{T}(I_{0,N}) \cap I_{0,N}) = \lambda(I_{0,N}) > 0$ , and, therefore,  $\tau(I_{0,N}) = 1$ .  $\square$

**Proposition 4.1.2.** *If the limit exists, then*

$$\rho(I_{i,N}) = - \lim_{n \rightarrow \infty} \frac{1}{n} \ln \frac{f_{i,N}^n}{2^n}.$$

*Proof.* Consider a Markov partition  $\mathcal{I}_N$  of a unit interval. There are  $f_{i,N}^n$  elements of this partition that do not enter the hole  $I_{i,N}$  in the first  $n$  iterations of  $T$ . The measure of each element is  $\frac{1}{2^{n+N}}$ . Thus  $1 - \lambda(\Omega_n(I_{i,N})) = \frac{f_{i,N}^n}{2^{n+N}}$ . Therefore,

$$\rho(I_{i,N}) = - \lim_{n \rightarrow \infty} \frac{1}{n} \ln (1 - \lambda(\Omega_n(I_{i,N}))) = - \lim_{n \rightarrow \infty} \frac{1}{n} \ln \frac{f_{i,N}^n}{2^n}.$$

$\square$

We now describe the distribution of periodic points among different holes. The following Proposition is a well known result(see, e.g. Proposition 1.7.2 in [25]).

**Proposition 4.1.3.** *The number,  $p(k)$ , of periodic points of period  $k$  (not necessary minimal) of the doubling map of the unit interval is equal to  $2^k - 1$  and the distance between two neighboring periodic points of the same period is equal to  $\frac{1}{2^k - 1}$ .*

In other words, periodic points of the same period are distributed uniformly in the unit interval.

**Corollary 4.1.4.** *An interval of the size  $\frac{1}{2^N}$ ,  $N \geq 1$ , contains at most one periodic point of a period  $k \leq N$ .*

**Corollary 4.1.5.** *Suppose  $x$  is a non-periodic point. Then for any positive integer  $n$  there exists  $\delta(x, n) > 0$  such that  $\delta$ -neighborhood of  $x$  does not contain any periodic points of periods less or equal  $n$ .*

**4.2. Symbolic Dynamics.** We can view all real numbers between 0 and 1 as binary numbers represented by one-sided infinite sequences of zeros and ones. In that case, the result of applying a doubling map is a number whose binary representation is obtained from the original one by erasing the first symbol and leaving the rest unchanged. This allows us to introduce the symbolic dynamics for the map under study.

Let  $\Omega(m)$  be a finite alphabet (set of symbols) of size  $m$ . A word  $w$  is a sequence of symbols from  $\Omega(m)$  of a finite or infinite length,  $w = \{w_i\}_{i=1}^j$ ,  $w_i \in \Omega(m)$ . Let  $|w|$  denote the length of a word  $w$ . The set  $\Lambda_{\Omega(m)}^+$  consists of all one-sided infinite words, i.e  $\Lambda_{\Omega(m)}^+ = \{\{w_i\}_{i=1}^\infty : w_i \in \Omega(m)\}$ . For a word  $w = \{w_i\}_{i=1}^k$  of a finite length  $|w| = k$  and a positive integer  $n$  define a *cylinder set* in  $\Lambda_{\Omega(m)}^+$ ,

$$C_w(n) = \left\{ W \in \Lambda_{\Omega(m)}^+ : W_{n+i-1} = w_i, i = 1 \dots |w| \right\} \subset \Lambda.$$



Consider now a Bernoulli measure  $\hat{\lambda}$  with equal probabilities on the collection of cylinder sets and extend it to the  $\sigma$ -algebra generated by this collection. In particular, the measure of the cylinder  $C_w(1)$  equals  $\hat{\lambda}(C_w(1)) = m^{-k}$ .

The shift map of  $\Lambda$  into itself is defined as  $(\sigma(w))_i = w_{i+1}$ , i.e.  $\sigma$  drops the first symbol and shifts the whole sequence to the left. The shift map preserves the Bernoulli measure. Then the triplet  $(\Lambda_{\Omega(m)}^+, \sigma, \hat{\lambda})$  defines a measurable dynamical system.

The doubling map is metrically equivalent to this one-sided shift on the space of infinite binary ( $m = 2$ ) sequences (see, for example [27] for details). A Markov hole (see Section 4) of the size  $2^{-N}$  corresponds to a cylinder defined by a word  $w$  of the size  $N$  and located at the first position,  $C_w(1)$ . The periodic points for the doubling map correspond to periodic words in the symbolic space.

**4.3. Escape rate.** Suppose that Markov hole  $I_{i,N}$  corresponds to the cylinder  $C_w(1)$ ,  $|w| = N$ . Then the set of points that do not enter the hole during the first  $n$  iterations of the doubling map corresponds to the set of points in  $\Lambda_{\Omega(m)}^+$  that do not enter the  $C_w(1)$  after applying a shift map  $n$  times, i.e. infinite words that do not contain the word  $w$  in the first  $n + N$  positions. Let  $c_w(n + N)$  be the number of such infinite words,

$$c_w(k) = \text{card} \left\{ v \in \Lambda_{\Omega(m)}^+ : \sigma^i(v) \notin C_w(1), i = 1 \dots k - |w| + 1 \right\}.$$

Since the escape rate for the doubling map into the hole  $I_{i,N}$  equals to the escape rate for the shift map into a corresponding cylinder set  $C_w(1)$  we have that

$$\rho(I_{i,N}) = \rho(C_w(1)) = - \lim_{n \rightarrow \infty} \frac{1}{n} \ln \frac{c_w(n + N)}{2^{n+N}},$$

if the limit exists.

It is known (see [22]) that there exists a positive integer  $n_0$  such that for all  $n \geq n_0$  there exist some constants  $c_1$ ,  $c_2$ , and the constant  $\theta_w < 2$  that depends only on  $w$  (but not on  $n$ ) such that

$$(4.1) \quad c_1 \theta_w^n \leq c_w(n) \leq c_2 \theta_w^n.$$

**Lemma 4.3.1.** *The escape rate  $\rho(C_w(1))$  is well defined. Moreover,*

$$(4.2) \quad \rho(C_w(1)) = - \lim_{n \rightarrow \infty} \frac{1}{n} \ln \frac{c_w(n + N)}{2^n} = - \ln \frac{\theta_w}{2},$$

where  $\theta_w < 2$  is a constant that depends on the word  $w$ .

Our next goal is to determine how does  $c_w(k)$  depend on  $w$ .

**4.4. Combinatorics on words.** As we have seen above in order to compute escape rate we need to count the number of binary words of a fixed length that do not contain a certain subword,  $c_w(k)$ . We use some results from the theory of combinatorics on words to obtain this number.

In [22] Guibas and Odlyzko studied (introduced, according to Guibas and Odlyzko, by J. Conway) a function from the set of finite words to itself called an autocorrelation function,  $\text{corr}(w)$ . Let  $w$  be a binary word of the size  $k$ . Then the value  $\text{corr}(w) = [b_1 \dots b_k]$  is determined in the following way. Place a copy of the word  $w$  under the original and shift it to the right by  $l$  digits. If the overlapping parts match then  $b_{l+1} = 1$ . Otherwise we set  $b_l = 0$ . In particular, we always have  $b_1 = 1$

(since the word matches itself). For example, suppose that  $w = [10100101]$ . Then  $\text{corr}(w) = [10000101]$ .

Note that we can also view  $\text{corr}(w)$  as a binary number and slightly abusing notations we denote both, a binary word and a binary number,

$$\text{corr}(w) = b_1 \cdot 2^{k-1} + b_2 \cdot 2^{k-2} + \dots + b_k.$$

In the example above we have

$$\text{corr}([10100101]) = 1 \times 2^7 + 0 \times 2^6 + 0 \times 2^5 + 0 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0.$$

Similarly, we can define a correlation polynomial [22],

$$(4.3) \quad f_w(z) = b_1 \cdot z^{k-1} + b_2 \cdot z^{k-2} + \dots + b_k, \quad z \in \mathbb{C},$$

so that  $f_w(2) = \text{corr}(w)$ .

Autocorrelation function, among other things, describes periodicities in the word. Consider a cylinder set  $C_w(1)$  generated by the word  $w$  and suppose that  $C_w(1)$  contains a periodic point  $v = \{v_i\}_{i=1}^{+\infty} \in C_w(1)$  of period  $\ell$ ,  $\ell < |w|$ . If  $\text{corr}(w) = [b_1 \dots b_k]$  then  $b_{\ell+1} = 1$ . (The autocorrelation function was used in [26] for computing of the dynamical Zeta function of subshifts of finite type.)

In order to compute the number of words of the size  $n$  avoiding a given word  $w$  of a length  $k$ ,  $c_w(n)$ , consider a generating function for  $c_w(n)$ ,

$$F_w(z) = \sum_{n=0}^{\infty} c_w(n) z^{-n}.$$

It was shown in [22] that  $F_w(z)$  is a rational function and the following asymptotic estimates on  $c_w(n)$  were obtained.

**Lemma 4.4.1.** *Suppose that  $w$  and  $u$  are two words of the same length and  $\text{corr}(w) > \text{corr}(u)$ . Then,*

$$\lim_{n \rightarrow \infty} \frac{\ln c_w(n)}{n} > \lim_{n \rightarrow \infty} \frac{\ln c_u(n)}{n}.$$

The following non-asymptotic result relating the number of sequences that do not contain certain word to its autocorrelation function was proved in [18].

**Lemma 4.4.2.** *Suppose that  $w$  and  $u$  are two words of the same length and  $\text{corr}(w) > \text{corr}(u)$ . Then there exists  $\tilde{n}_0$ , such that for all  $n \geq \tilde{n}_0$  one has that*

$$(4.4) \quad c_w(n) > c_u(n).$$

Then in [13] and [30] this result was improved by finding an explicit value of  $\tilde{n}_0$  and expanding the result to the words of different lengths and to the systems with the alphabet of any finite size. Specifically, consider binary words of equal length,  $w$  and  $u$ . Suppose that  $\text{corr}(w) = [b_1 \dots b_N]$  and  $\text{corr}(u) = [a_1 \dots a_N]$ . Then,

$$(4.5) \quad \tilde{n}_0 = N + \min\{i : b_i \neq a_i\} - 1.$$

**4.5. Main result.** Lemma 4.4.2 leads to the following relationship between the correlation function on one hand and the escape rate into and survival probability of the cylinder generated by the corresponding word on the other hand.

**Lemma 4.5.1.** *Suppose that  $w$  and  $u$  are two words of the same length. Let  $C_w(1)$  and  $C_u(1)$  be two cylinder sets generated by these two words. Then*

$$\text{corr}(w) > \text{corr}(u) \Rightarrow \rho(C_w(1)) < \rho(C_u(1)).$$

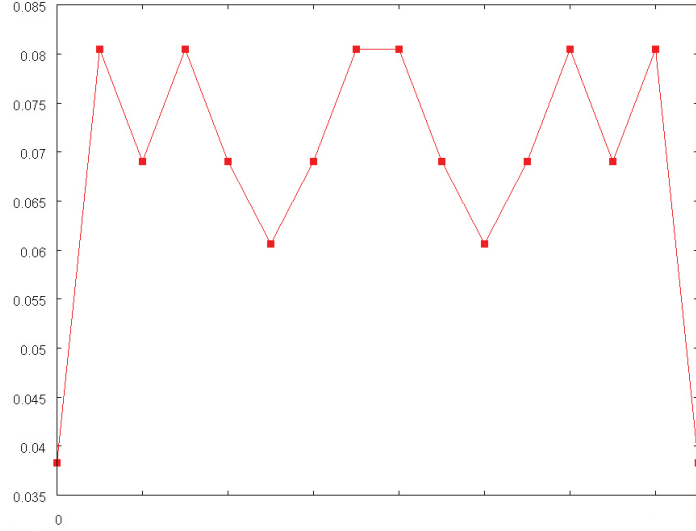


FIGURE 1. Escape rate vs. position of the Markov hole of size  $2^{-N}$ ,  $N = 4$ .

By making use of equations 4.4 and 4.5 we immediately obtain

**Lemma 4.5.2.** *Suppose that  $w$  and  $u$  are two words of the same length with  $\text{corr}(w) > \text{corr}(u)$ . Let  $\text{corr}(w) = [b_1 \dots b_N]$  and  $\text{corr}(u) = [a_1 \dots a_N]$ . Let  $C_w(1)$  and  $C_u(1)$  be two cylinder sets generated by these two words. Then for all  $n \geq \min\{i : b_i \neq a_i\} - 1$ , one has  $\lambda(\Omega_n(C_w(1))) < \lambda(\Omega_n(C_u(1)))$ .*

Finally, we turn to the proof of the main theorems of this section, which state that the escape rate is larger for the hole that has a larger Poincaré recurrence time (asymptotic result). Moreover, the survival probability is smaller for the hole that has a larger Poincaré recurrence time (non-asymptotic result).

*Proof of Theorem 4.0.8.* Let  $w$ ,  $|w| = N$ , be the word that codes the hole  $I_{i,N}$  and let  $C_w(1)$  be the cylinder generated by that word. Similarly, let  $u$ ,  $|u| = N$ , be the word that codes the hole  $I_{j,N}$  and let  $C_u(1)$  be the cylinder generated by that word. Consider the autocorrelation functions of  $w$  and  $u$ ,  $\text{corr}(w) = [b_1 \dots b_N]$  and  $\text{corr}(u) = [a_1 \dots a_N]$ . As we have seen before,  $b_1 = a_1 = 1$ . Also,  $b_{\ell+1} = 1$  if and only if  $C_w(1)$  contains a periodic point of period  $\ell$ ,  $1 \leq \ell < N$ . The same is true for  $a_{\ell+1}$ . Hence, the first non-zero element of  $[b_1 \dots b_N]$  after  $b_1$  is  $b_{\tau(I_{i,N})+1}$ . Thus, if  $\tau(I_{j,N}) > \tau(I_{i,N})$  then  $\text{corr}(u) < \text{corr}(w)$ . But by Lemma 4.5.1,  $\rho(I_{j,N}) = \rho(C_u(1)) > \rho(C_w(1)) = \rho(I_{i,N})$ . From Lemma 4.5.2 one has that  $\min\{i : b_i \neq a_i\} = \tau(I_{i,N}) + 1$ . Since  $\lambda(\Omega_n(I_{j,N})) = \lambda(\Omega_n(C_u(1)))$  and  $\lambda(\Omega_n(I_{i,N})) = \lambda(\Omega_n(C_w(1)))$ , the inequality  $1 - \lambda(\Omega_n(I_{j,N})) < 1 - \lambda(\Omega_n(I_{i,N}))$  holds for all  $n \geq \tau(I_{i,N})$ .  $\square$

**4.6. Local escape rate.** Our next task is to investigate what happens to the escape rate as the size of the hole is decreasing. Let  $I_N(x)$  be an element of a partition  $\mathcal{I}_N$  (see Section 4.1) that contains a point  $x$ .

For a large class of symbolic systems, which includes the expanding maps of the interval considered here, for almost every point Poincaré recurrence time grows

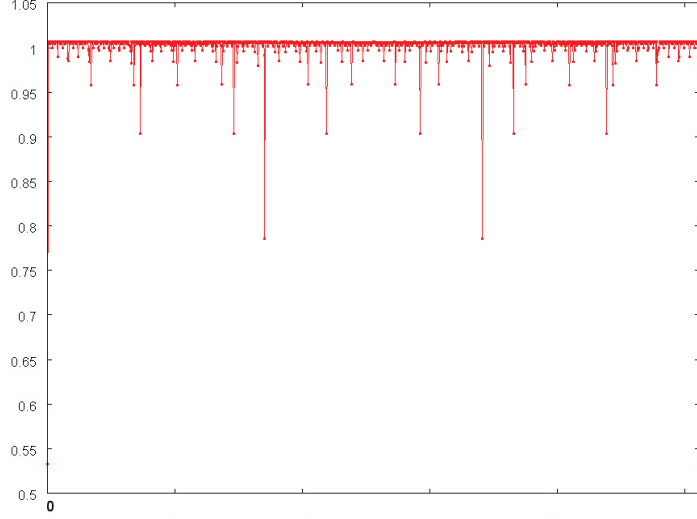


FIGURE 2. Plot of  $\frac{\rho(I_N(x))}{\lambda(I_N(x))}$  vs  $x$ ,  $N = 10$ .

linearly with  $N$  as a size of the hole exponentially decreases,  $\lim_{N \rightarrow \infty} \frac{\tau(I_N(x))}{N} = 1$  [3]. The following theorem provides a similar result for the escape rate.

**Theorem 4.6.1.** *Consider the doubling map of the unit interval,  $x \in [0, 1]$ .*

a) *If  $x$  is a periodic point of period  $m$  then*

$$\lim_{N \rightarrow \infty} \frac{\rho(I_N(x))}{\lambda(I_N(x))} = 1 - \frac{1}{2^m}.$$

b) *If  $x$  is a non-periodic point then*

$$\lim_{N \rightarrow \infty} \frac{\rho(I_N(x))}{\lambda(I_N(x))} = 1.$$

*Proof.* Using equality 4.2 and the fact that  $\lambda(I_N(x)) = 2^{-N}$ , we obtain,

$$\lim_{N \rightarrow \infty} \frac{\rho(I_N(x))}{\lambda(I_N(x))} = - \lim_{N \rightarrow \infty} 2^N \ln \frac{\theta_w}{2},$$

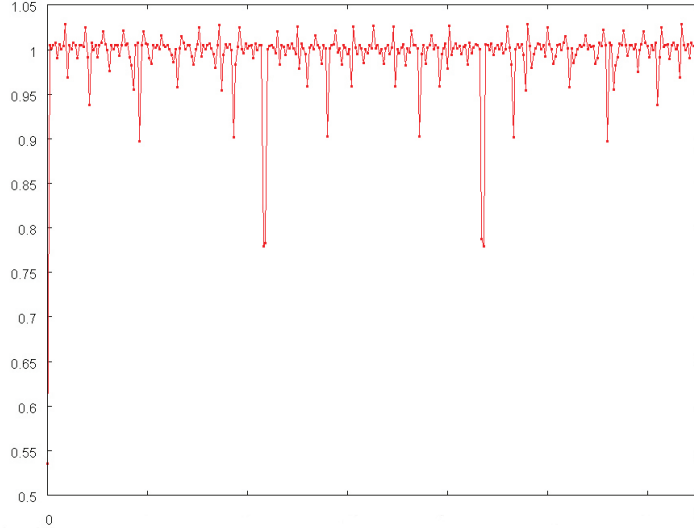
where  $\theta_w$  depends on  $N$ .

It was shown in [20] that asymptotically the constant  $\theta_w$  satisfies  $\ln \theta_w = \ln 2 - \frac{1}{2f_w(2)} + \mathcal{O}(2^{-2N})$ , where  $f_w(z)$  is a correlation polynomial 4.3. Suppose that  $x$  is non-periodic point. Then one has that,

$$\lim_{N \rightarrow \infty} 2^{-N} \sum_{i=\tau(I_N(x))}^N b_i(N) 2^{N-i} = 0.$$

Therefore,

$$\lim_{N \rightarrow \infty} \frac{\rho(I_N(x))}{\lambda(I_N(x))} = \lim_{N \rightarrow \infty} \frac{2^N}{2^N + \sum_{i=\tau(I_N(x))}^N b_i(N) 2^{N-i}} = 1.$$

FIGURE 3. Plot of  $\frac{\rho(A(x))}{\lambda(A(x))}$  vs.  $x$ ,  $\lambda(A(x)) = \frac{1}{327}$ .

Let now  $x$  be a periodic point of (minimum) period  $m$ . Then,

$$\lim_{N \rightarrow \infty} 2^{-N} \sum_{i=\tau(I_N(x))}^N b_i(N) 2^{N-i} = \sum_{k=1}^{\infty} 2^{-km} = \frac{1}{1-2^{-m}} - 1.$$

Therefore,

$$\lim_{N \rightarrow \infty} \frac{\rho(I_N(x))}{\lambda(I_N(x))} = \lim_{N \rightarrow \infty} \frac{2^N}{2^N + \sum_{i=\tau(I_N(x))}^N b_i(N) 2^{N-i}} = 1 - \frac{1}{2^m}.$$

□

Suppose that an interval  $A$  does not contain points  $x = s2^{-k}$  for all  $k \leq n$  for some  $n > 1$ ,  $s, k, n \in \mathbb{N}$ , i.e.  $A$  does not contain an end point of an element of any Markov partition  $\mathcal{I}_N$ ,  $N \leq n$ . Then one can find two elements of the Markov partitions  $I_{N_1}(x)$  and  $I_{N_2}(x)$  so that  $I_{N_1}(x) \subseteq A \subseteq I_{N_2}(x)$ . This implies Theorem 4.0.9.

**Remark 4.6.2.** Observe that in Fig. 3 there are many local maxima of  $\frac{\rho(A(x))}{\lambda(A(x))}$  which are essentially larger than one. These happen at the places where the intervals  $A(x)$  have maximum Poincaré return time among all such intervals  $A(x)$  with length of  $A(x)$  being fixed. Indeed, consider a set of holes that are Markov, say  $\mathcal{I}_N$ . The ratio  $\frac{\rho(I_{i,N})}{\lambda(I_{i,N})}$  attains its maximum (see Corollary 4.7.1 below) in the holes that have the largest Poincaré return time (in this case  $N$ ). Now, suppose that  $\lambda(I_{i,N+1}) \leq \lambda(A(x)) \leq \lambda(I_{i,N})$ ,  $A(x) \subset I_{i,N}$ , and  $\tau(A(x)) = \tau(I_{i,N})$ . Then  $\rho(A(x)) \approx \rho(I_{i,N})$ , but  $\frac{\rho(A(x))}{\lambda(A(x))} \geq \frac{\rho(I_{i,N})}{\lambda(I_{i,N})}$ . Thus, by shrinking a Markov hole while keeping the same Poincaré return time we can increase the ratio of the escape rate to the size of the hole.

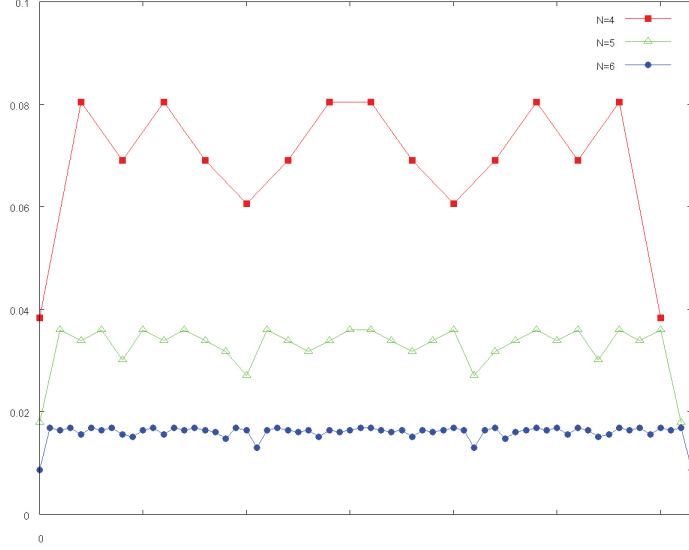


FIGURE 4. Escape rate vs. position of the Markov hole of size  $2^{-N}$ .

**4.7. More results for one hole.** In the case of the doubling map with Markov holes we know precisely where to make a hole to achieve maximum (or minimum) escape rate.

**Corollary 4.7.1.**

$$\min_{1 \leq i \leq 2^N} \rho(I_{i,N}) = \rho(I_{1,N}), \quad \max_{1 \leq i \leq 2^N} \rho(I_{i,N}) = \rho(I_{2,N}),$$

although the holes that give these extremes are not unique.

*Proof.* By Proposition 4.1.3, one has that  $1 \leq \tau(I_{i,N}) \leq N$ . It is easy to check that  $\tau(I_{1,N}) = 1$  and  $\tau(I_{2,N}) = N$ . The result follows from the Main Theorem.  $\square$

Note that we obtained minimum escape rate in one more interval that contain a fixed point, namely  $I_{2^N,N}$ . The maximum is obtained in the intervals that have a minimum period equal to  $N$ , as Figure 1 illustrates this.

Next we prove Theorem 4.0.10 that states that the escape rate decreases monotonically as we decrease the size of the hole.

*Proof of Theorem 4.0.10.* It follows from Corollary 4.7.1 that  $\max_{1 \leq i \leq 2^{N+1}} \rho(I_{i,N+1}) = \rho(I_{2,N+1})$  and  $\min_{1 \leq i \leq 2^N} \rho(I_{i,N}) = \rho(I_{1,N})$ . Let  $w_1$  and  $w_2$  be two binary words that define holes  $I_{2,N+1}$  and  $I_{1,N}$ , respectively. then  $|w_1| = N + 1$  and  $|w_2| = N$ . It is easy to check that  $w_1 = \underbrace{0 \dots 0}_N 1$  and  $w_2 = \underbrace{0 \dots 0}_N$ .

Let  $c_2$  be the number of words of length  $n \gg N$  that have  $N$  consecutive zeros and let  $c_1$  be the number of words of length  $n$  that have  $N$  consecutive zeros followed by a one. Clearly  $c_2 > c_1$ . On the other hand, if a word has  $N$  zeros, then that sequence of zeros will either eventually terminate with a one, or else all the remaining symbols in the entire word of length  $n$  are zeros. The number of latter

words is obviously smaller than  $c_1$ , that is  $c_2 - c_1 < c_1$ . Hence,  $c_1$  and  $c_2$  have the same asymptotic as  $n \rightarrow \infty$ . This implies the statement of the theorem.  $\square$

The next result deals with arbitrary (not necessarily the elements of Markov partition) but sufficiently small holes and follows directly from Corollary 4.0.9.

**Theorem 4.7.2.** *Suppose  $x_1$  and  $x_2$  are two periodic points with periods  $m_1$  and  $m_2$ , respectively,  $m_1 < m_2$ . Then there exists a positive number  $\varepsilon_0 = \varepsilon_0(m_1, m_2)$  such that for all subintervals  $A_1$  and  $A_2$  of the unit interval with  $x_i \in \text{int}(A_i)$ ,  $i = 1, 2$ ,  $\lambda(A_1) = \lambda(A_2) < \varepsilon_0$ , one has that  $\rho(A_1) < \rho(A_2)$ .*

#### 4.8. Contributions to the escape rate: the size of the hole vs. dynamics.

The doubling map shows that the escape rate is not determined by the size of the hole alone. The following example demonstrates the possibility of a larger escape rate into the smaller hole.

Consider two holes  $A \cup B$  and  $C$ , where  $A = [0, \frac{1}{4})$ ,  $B = [\frac{1}{4}, \frac{5}{16})$ , and  $C = [\frac{1}{2}, \frac{3}{4})$ . Note that  $B \subset \hat{T}^{-2}A$ . Clearly,  $\lambda(A \cup B) = \frac{5}{16} > \lambda(C) = \frac{4}{16}$ . It is easy to check that  $\tau(A) = 1$  and  $\tau(C) = 3$ , thus  $\rho(A) < \rho(C)$ . On the other hand, by Proposition 2.3.2 it follows that  $\rho(A) = \rho(A \cup B)$ . Therefore we have that  $\rho(A \cup B) < \rho(C)$ . Hence,  $\rho([0, \frac{5}{16})) < \rho([\frac{1}{2}, \frac{3}{4}))$ , but  $\lambda([0, \frac{5}{16})) > \lambda([\frac{1}{2}, \frac{3}{4}))$ .

In general, the following result shows that there are holes of the arbitrarily large size with arbitrarily small escape rate.

**Theorem 4.8.1.** *For any  $\varepsilon \in (0, 1)$  and any  $r > 0$  there exists a measurable set  $A \subset [0, 1]$  such that*

$$\lambda(A) > 1 - \varepsilon, \quad \rho(A) < r.$$

*Proof.* We can always pick  $N$  such that  $\rho(I_{i,N}) < r$  for some  $i$ . One has that for any finite  $n$ ,  $\lambda(\bigcup_{j=0}^n \hat{T}^{-j} I_{i,N}) < 1$ , but  $\lambda(\bigcup_{j=0}^{\infty} \hat{T}^{-j} I_{i,N}) = 1$ . Thus for any  $\varepsilon > 0$  we can find  $n_0$  such that  $1 > \lambda(\bigcup_{j=0}^{n_0} \hat{T}^{-j} I_{i,N}) > 1 - \varepsilon$ . By Proposition 2.3.2 one has that  $\rho(I_{i,N}) = \rho(\bigcup_{j=0}^{n_0} \hat{T}^{-j} I_{i,N})$ . Now, set  $A = \bigcup_{j=0}^{n_0} \hat{T}^{-j} I_{i,N}$ .  $\square$

## 5. SOME GENERALIZATIONS.

**5.1. Linear expanding map.** As was mentioned at the beginning of this section the same results hold for the map  $x \mapsto \kappa x \pmod{1}$ ,  $\kappa \in \mathbb{N}$  and  $\kappa > 1$ .

**5.2. Tent map.** The tent map is a map  $\hat{T}$  of a unit interval to itself given by

$$\hat{T}(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2} \\ 2 - 2x, & \frac{1}{2} \leq x \leq 1 \end{cases}.$$

This map preserves a Lebesgue measure on  $[0, 1]$ . We can use the following correspondence between the tent map and the symbolic dynamics:  $s_n = 0$  if  $\hat{T}^n x < 0.5$  and  $s_n = 1$  otherwise. It can be easily shown that mapping  $x \mapsto \{s_n\}$  is a metric conjugacy onto the left shift symbolic space. Then one can repeat all the arguments that we used for the doubling map to obtain similar result. Namely, for  $I_{i,N}$  defined as before,  $I_{i,N} = [\frac{i-1}{2^N}, \frac{i}{2^N}]$ ,  $i = 1 \dots 2^N$ , the following statement holds.

**Theorem 5.2.1.** *Consider a tent map.  $\forall N \in \mathbb{N}$  and  $\forall i, j = 1, \dots, 2^N$  we have that if  $\tau(I_{j,N}) > \tau(I_{i,N})$  then  $\rho(I_{j,N}) > \rho(I_{i,N})$ .*

**5.3. Logistic map.** Consider a logistic map  $\hat{T} : [0, 1] \rightarrow [0, 1]$  given by

$$\hat{T}(x) = 4x(1-x), \quad x \in [0, 1].$$

It preserves a measure,  $\lambda$ , with the density  $\frac{1}{\pi\sqrt{x(1-x)}}$ .

A logistic map and the tent map are metrically conjugate: it is easy to check that conjugacy is given by the transformation  $y = \sin^2 \frac{\pi x}{2}$ . Thus, by Lemma 2.3.4, for  $I_{i,N}$  (preimages of a finite Markov partition for  $\hat{T}$ ) defined as  $I_{i,N} = \left[ \frac{2}{\pi} \arcsin \sqrt{\frac{i-1}{2^N}}, \frac{2}{\pi} \arcsin \sqrt{\frac{i}{2^N}} \right]$ ,  $i = 1 \dots 2^N$ , the following theorem holds.

**Theorem 5.3.1.** *Consider a logistic map. Suppose that  $\tau(I_{j,N}) > \tau(I_{i,N})$ . Then,  $\rho(I_{j,N}) > \rho(I_{i,N})$ .*

**5.4. Baker's map.** A Baker's map is an example of two dimensional invertible hyperbolic chaotic map. This map,  $\hat{T} : [0, 1] \times [0, 1] \rightarrow [0, 1] \times [0, 1]$ , is defined as

$$\hat{T}(x, y) = \left( 2x \mod 1, \frac{1}{2}(y + \lfloor 2x \rfloor) \mod 1 \right),$$

where  $\lfloor x \rfloor$  is an integer part of  $x$ . It preserves the Lebesgue measure and is ergodic.

It was observed numerically in [34] that escape rate in this system depends on the position of the hole. The Markov holes for this map are the rectangles given by

$$I_{i,j,N,M} = \left[ \frac{i-1}{2^N}, \frac{i}{2^N} \right] \times \left[ \frac{j-1}{2^M}, \frac{j}{2^M} \right], \quad i = 1 \dots 2^N, \quad j = 1 \dots 2^M.$$

**Theorem 5.4.1.** *Consider the Baker's map  $\hat{T}$ . Then  $\forall N, M \in \mathbb{N}$ ,  $\forall i_1, i_2 = 1, \dots, 2^N$ , and  $\forall j_1, j_2 = 1, \dots, 2^M$  we have that if  $\tau(I_{i_1,j_1,N,M}) > \tau(I_{i_2,j_2,N,M})$  then  $\rho(I_{i_1,j_1,N,M}) > \rho(I_{i_2,j_2,N,M})$ .*

The statement follows from the conjugacy of Baker's map and the full binary shift.

## 6. CONCLUSIONS.

Apparently there are still other natural and interesting unexplored questions on the dynamics of open dynamical systems. In this paper we dealt with one such problem, the dependence of the escape rate on the position of a hole. We demonstrated that dynamical characteristics can play as important role for the escape rate as the size (measure) of the hole does.

Here we just scratch the surface of this new area in the theory of open dynamical systems dealing with the uniformly hyperbolic systems. In general, the effect of distortion (variability of values of the derivative or Jacobian of the dynamical system) can also essentially contribute to the escape through a hole of a given size. This problem will be addressed in another paper.

## ACKNOWLEDGEMENTS.

Authors are grateful to V. Afraimovich, M. Demers, M. Pollicott, and T. Tél for numerous helpful discussions. Authors also thankful to the anonymous referee for helpful comments and suggesting a much shorter and an elegant proof of Theorem 4.0.10.



## REFERENCES

- [1] V. Afraimovich, Pesin's dimension for Poincaré recurrences, *Chaos*, **7** (1997), 12–20.
- [2] V. Afraimovich, J-R. Chazottes, and B. Saussol, Local dimensions for Poincaré recurrences, *Electronic Research Announcements of AMS*, **6** (2000), 64–74.
- [3] V. Afraimovich, E. Ugalde and J. Urias, *Fractal dimensions for Poincaré recurrences*, Elsevier Science, 2006.
- [4] E. G. Altmann and T. Tél, Poincaré recurrences from the perspective of transient chaos, *Phys Rev Lett.* **17** (2008), 174101.
- [5] V. Baladi, C. Bonatti, and B. Schmitt, Abnormal escape rates from nonuniformly hyperbolic sets, *Ergod. Th. and Dynam. Sys.*, **19** (1999), 1111–1125.
- [6] G. E. Bredon, *Topology and Geometry*, Springer, 1993.
- [7] H. Bruin, M. Demers, and I. Melbourne, Existence and convergence properties of physical measures for certain dynamical systems with holes, preprint (2007).
- [8] H. Buljan and V. Paar, Bursts in the chaotic trajectory lifetimes preceding controlled periodic motion, *Phys. Rev. E*, **62** (2000), 4869–4872.
- [9] H. Buljan and V. Paar, Many-hole interaction and average lifetimes of chaotic transients that precede controlled periodic motion, *Phys. Rev. E*, **63** (2001), 066205.
- [10] N. Chernov and R. Markarian, Ergodic properties of Anosov maps with rectangular holes, *Bol. Soc. Bras. Mat.*, **28** (1997), 271314.
- [11] N. Chernov, R. Markarian, and S. Troubetzkoy, Conditionally invariant measures for Anosov maps with small holes, *Ergod. Th. and Dynam. Sys.*, **18** (1998), 1049–1073.
- [12] N. Chernov, R. Markarian, and S. Troubetzkoy, Invariant measures for Anosov maps with small holes, *Ergod. Th. and Dynam. Sys.*, **20** (2000), 1007–1044.
- [13] I. Cakir, O. Chryssaphinou, and M. Månsson, On a conjecture by Eriksson concerning overlap in strings, *Combin. Probab. Comput.*, **8** (1999), 429–440.
- [14] P. Collet, S. Martinez, and B. Schmitt, The Yorke-Pianigiani measure and the asymptotic law on the limit Cantor set of expanding systems, *Nonlinearity*, **7** (1994), 1437–1443.
- [15] M. Demers and L.-S. Young, Escape rates and conditionally invariant measures, *Nonlinearity*, **19** (2006), 377–397.
- [16] M. Demers, Markov extensions and conditionally invariant measures for certain logistic maps with small holes, *Ergod. Th. and Dynam. Sys.*, **25:4** (2005), 1139–1171.
- [17] M. Demers, Markov extensions for dynamical systems with holes: an application to expanding maps of the interval, *Israel J. of Math.*, **146** (2005), 189–221.
- [18] K. Eriksson, Autocorrelation and the enumeration of strings avoiding a fixed string, *Combin. Probab. Comput.*, **6** (1997), 45–48.
- [19] M. J. Erickson, *Introduction to combinatorics*, A Wiley-Interscience Publication, 1996.
- [20] L. J. Guibas and A. M. Odlyzko, Maximal Prefix-Synchronized Codes, *SIAM Journal on Applied Mathematics*, **35** (1978), 401–418.
- [21] L. J. Guibas and A. M. Odlyzko, Periods in strings, *J. Combin. Theory Ser. A*, **30** (1981), 19–42.
- [22] L. J. Guibas and A. M. Odlyzko, String overlaps, pattern matching, and nontransitive games, *J. Combin. Theory Ser. A*, **30** (1981), 183–208.
- [23] V. Halava, T. Harju and L. Ilie, Periods and binary words, *J. Comb. Theory Ser. A*, **89** (2000), 298–303.
- [24] O. Jenkinson, Maximum hitting frequency and fastest mean return time, *Nonlinearity*, **18** (2005), 2305–2321.
- [25] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, 1998.
- [26] D. Lind, Perturbation of Shifts of Finite Type, *SIAM J. Disc. Math.*, **2** (1989), 350–365.
- [27] D. Lind and B. Marcus, *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press, 1995.
- [28] C. Liverani and V. Maume-Deschamps, Lasota-Yorke maps with holes: conditionally invariant probability measure and invariant probability measures on the survivor set *Ann. Inst. H. Poincaré' Probab. Statist.* **39** (2003), no. 3, 385–412.
- [29] M. Lothaire, *Applied Combinatorics on Words*, Cambridge University Press, 2005.
- [30] M. Månsson, Pattern avoidance and overlap in strings, *Combin. Probab. Comput.*, **11** (2002), 393–402.

- [31] V. Paar and N. Pavin, Discontinuous pattern of cumulative separation between orbits of chaotic transient, *Phys. Lett. A*, **235** (1997), 139–144.
- [32] V. Paar and N. Pavin, Bursrs in average lifetime of transients for chaotic logistic map with a hole, *Phys. Rev. E*, **55** (1997), 4112–4115.
- [33] G. Pianigiani and J. Yorke, Expanding maps on sets which are almost invariant: decay and chaos, *Trans. AMS*, **252** (1979), 351–366.
- [34] J. Schneider, T. Tél, and Z. Neufeld, Dynamics of "leaking" Hamiltonian systems, *Phys. Rev. E*, **66** (2002), 2181–2186.
- [35] H. Van den Bedem and N. Chernov, Expanding maps of interval with holes, *Ergod. Th. and Dynam. Sys.*, **22** (2002), 637–654.
- [36] P. Walters, *An Introduction to Ergodic Theory*, Springer, 1982.

SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA, GA 30332  
*E-mail address*: `bunimovh@math.gatech.edu`

SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA, GA 30332  
*E-mail address*: `alex@yurchenko.org`