DETERMINISTIC WALKS IN RIGID ENVIRONMENTS WITH AGING.

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Abstract. Aging is an abundant property of materials, populations, and networks. We consider some classes of cellular automata (Deterministic Walks in Random Environments) where the process of aging is described by a time dependent function, called a rigidity of the environment. Asymptotic laws for the dynamics of perturbations propagating in such environments with aging are obtained.

1. Introduction

It is well known that properties of materials, cells, species, environments, etc, do change with time ([1]-[5], [10]-[14]). This process usually called an *aging*, is universal in natural as well as in artificial systems. It involves any individual subsystems as well as their networks, populations, etc.

In this paper we consider a new class of models with aging. This class is rather rich and it allows for efficient numerical simulations as well as for analytical studies, where exact formulas could be obtained. These systems belong to a very broad class of cellular automata called deterministic walks in random environments [8].

Deterministic walks in random environments (DWRE) are discrete in time systems on any graph $\mathcal{G}$ which describe a motion of an object (in what follows we call it a particle) that jumps between vertices of $\mathcal{G}$. We assume that graph $\mathcal{G}$ is a lattice, i.e. a non-directed graph, and edges of $\mathcal{G}$ are line segments of the length one. The particle moves between the neighboring vertices of the graph on these straight segments. By the *structure of an orbit* we mean a sequence of such line segments. If the particle occupies a vertex g then a choice of the next vertex $\tilde{g}$ is completely determined by a *scatterer* (local scattering rule) $S(g,t)$ which currently occupies the vertex g and by *edge* along which the particle came to g, where $t \in \mathbb{N}$ is discrete time.

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There is an initially prepared deterministic program at each vertex $g \in \mathcal{G}$, which is formed by a fixing from the very beginning a family of sequences of scatterers $S(g,t) = \{S_i(g,t)\}_{i=1}^{\infty}$. The dynamics of DWRE is defined as following: when the object comes to a vertex g at the first time it gets scattered by the scatterer $S_1(g,t)$; when it returns to g again it gets scattered by the scatterer $S_2(g,t)$ and so on, i.e. after the n-th visit to g the object gets scattered by the scatterer $S_n(g,t)$. The collection of all scatterers in all vertices of $\mathcal{G}$ form an (instant) *environment* in which the particle moves. Two types of environments are possible: fixed environment and evolving environment. In the *fixed environment* every time the particle arrives at a vertex g it encounters one and the same scatterer. On the other hand, in the *evolving environment* the type of the scatterer at the vertex g can change upon visits by the particle to the vertex. There are two possibilities: either the particle first gets scattered then changes the scatterer or changes the scatterer and then gets scattered by the new scatterer. The dynamics is similar in both cases so we use the first option.

One can contrast this dynamics with random walks where a moving object at each vertex performs a random trial ("throws a dice") to choose an edge along which to continue the motion. In DWRE the evolution of the environment is deterministic, hence the choice of the next vertex is deterministic, not random. The randomness comes into play when we setup the model: at the time $t = 0$ the sequences of scatterers $S(g,t)$ are assumed to be randomly distributed over vertices of the graph $\mathcal{G}$.

For these models two major problems are of interest. At first, we study a geometric structure of a "typical" orbit, i.e. dynamical properties of a model. After that we introduce some initial distribution of scatterers and then investigate statistical properties of the ensemble of all orbits.

The structure of the paper is the following. In Section 2 we define Walks in Rigid Environments [6]. Section 3 reviews the necessary results on these models without aging [6] - [9]. Section 4 deals with the environments with aging. The last Section 5 contains some concluding remarks.

2. Walks in rigid environments

There is not much hope to rigorously study DWRE with an arbitrary program (sequences of scatterers) allowed at a vertex. Therefore we need to identify some sufficiently general classes of DWRE that allow rigorous analysis. One such class of DWRE has been introduced in [6].

In these models an environment is characterized by a function

$$r : \mathcal{G} \times \mathbb{N} \to \mathbb{N},$$

called *rigidity*, that describes the sequence of scatterers $\{S_i(g,t)\}$ that are located on each vertex: a type of scatterer at the vertex g changes after the $r(g,t)$th visit by the particle to z. For example, if rigidity is a constant function, $r(g,t) = r$, the family of sequences of scatterers that sits on each vertex has the following structure:

$$S(g,t) = S(g) = \{\underbrace{S^1(g), S^1(g), \ldots, S^1(g)}_{r \text{ times}}, \underbrace{S^2(g), S^2(g), \ldots, S^2(g)}_{r \text{ times}}, \ldots\},$$

where $S^i(g)$ is some type of a scatterer that is allowed on a vertex g.

In the present paper we study Walks in Rigid Environments with *aging*, i.e. rigidity function is not a constant. Rather the rigidity function is a piecewise constant in time independent of the vertex function defined as

$$r(g,t) = r(t) = r_j,$$
$$j \in \mathbb{N}, \quad \forall g \in \mathcal{G}, \quad t \in [\tau_{j-1}, \tau_j) \cap \mathbb{N},$$

for some $\tau_0 = 0 \leq \tau_1 \leq \tau_2 \leq \ldots$, $\tau_j \in \mathbb{N}$, $r_j \in \mathbb{N}$.

Let $\eta(z,t)$, called an *index of a scatterer* at the site g at the time t, be the number of visits of the particle to a site g, which occurred between the last moment of time when scatterer at g changed type and t. Then the type of a scatterer at a vertex $g \in \mathcal{G}$ changes at the moment of time when the index of a scatterer at the vertex g, $\eta(g,t)$, is equal to the rigidity, $r(t)$, that is at the smallest time t_0 such that $r(t_0) = \eta(g, t_0)$.

Walks in the environment with a constant rigidity were studied in [6] - [9].

In this paper we study the models on $\mathbb{Z}^1$. For $\mathbb{Z}^1$ there are only four types of scatterers: the left scatterer ($'\texttt{LS}'$), the right scatterer ($'\texttt{RS}'$), the backward scatterer ($'\texttt{BS}'$), and the forward scatterer ($'\texttt{FS}'$). The reflection is the only nontrivial symmetry of $\mathbb{Z}^1$. Two of the scatterers are invariant under reflection ($'\texttt{BS}'$ and $'\texttt{FS}'$) and two are not ($'\texttt{LS}'$ and $'\texttt{RS}'$). Therefore, it is natural to consider two types of models ([6], [7]), each of them having only two types of scatterers, call them S_1 and S_2. The first one with $S_1 = {}'\texttt{FS}'$ and $S_2 = {}'\texttt{BS}'$ we will call NOS-model (model with non-oriented scatterers). The second with $S_1 = {}'\texttt{LS}'$ and $S_2 = {}'\texttt{RS}'$ we will call OS-model (model with oriented scatterers).

We assume that in each model initially the scatterers distributed identically and independently among vertices: the probability of seeing a scatterer S_1 at time zero at any site is q and the probability of seeing a scatterer S_2 at the same site is $1-q$. The case where scatterers form a Markov chain was considered in [9].

Denote by Ω a collection of all initial configurations of scatterers, $\Omega = \{S_1, S_2\}^{\mathbb{Z}}$. Let $\omega \in \Omega$ be some fixed initial configuration of scatterers and let $\omega(z,t)$ be the type of a scatterer at the vertex z at the time t.

Definition 2.0.1. *We say that a configuration of scatterers ω has a* positive tail *of a scatterer S if there exists an integer $n > 0$ such that all sites $z > n$ in the configuration are occupied by the scatterer S.*

We can similarly define a negative tail.

Clearly, the set of all configurations with a positive and/or a negative tail form a set (in the collection of all configurations) of measure zero.

Let $z(t)$ and $v(t)$ be a position and a velocity of the particle at the time t when it arrives at the vertex and let $v(t_+)$ be a velocity of the particle at time t right after it is changed by the scatterer located at $z(t)$. Denote by $z_{max}(t)$ and $z_{min}(t)$ positions of the largest and and smallest sites that particle visited during the time interval $[0,t]$. The velocity of a particle has only two possible values, $v(t) = \pm 1$. Without any loss of generality we can assume that initially particle moves to the right, i.e. $v(0) = +1$.

In this notation four possible scatterers are then characterized by the following relationships:

$$\omega(z,t) = '\mathtt{BS}' \Rightarrow v(t_+) = -v(t), \quad \omega(z,t) = '\mathtt{FS}' \Rightarrow v(t_+) = v(t),$$
$$\omega(z,t) = '\mathtt{RS}' \Rightarrow v(t_+) = +1, \quad \omega(z,t) = '\mathtt{LS}' \Rightarrow v(t_+) = -1.$$

3. Models with constant rigidity on $\mathbb{Z}^1$

Let us first consider the case of constant rigidity, say $r(z,t) = r, \forall z \in \mathbb{Z}, \forall t \in \mathbb{N}$. Assume that initially $(t = 0)$ the particle was at the origin, $z(0) = 0$. The following results were obtained in [6]-[8].

3.1. **OS model.**

The following theorem characterizes a geometric structure of a "typical" orbit. It states that almost all orbits oscillate.

Theorem 3.1.1. *For almost all initial configurations of scatterers the particle will visit each site of the lattice infinitely many times as $t \to +\infty$.*

The set of initial configurations for which the theorem fails consists of all the configurations with the positive tail of right scatterers or negative tail of left scatterers. This set has a zero measure in the set of all initial configurations.

The next theorem describes the statistical properties of the OS model.

Theorem 3.1.2. *For all values of $t > 0$ and all finite values of rigidity r, $\mathbb{E}z(t) = 0$. Moreover, there exist $T = T(q, r) \in \mathbb{N}$, $C_1 = C_1(q) > 0$, and $C_2 = C_2(q) > 0$ such that*

$$C_1 \frac{t}{r} \leq \mathbb{E}z^2(t) \leq C_2 \frac{t}{r}, \quad \forall t \geq T.$$

3.2. **NOS model.** In this model the dynamics is determined by the parity of the rigidity: if rigidity is even, then almost all trajectories are oscillatory and if rigidity is odd, then all orbits eventually propagate in one direction with constant average velocity.

The first theorem shows the geometric structure of the orbits.

Theorem 3.2.1. *If the value of the rigidity r is even, then for almost all initial configurations of scatterers the particle will visit each site of the lattice infinitely many times.*

On the other hand, if the value of the rigidity r is odd then for all initial configurations of scatterers the particle is eventually propagating into one direction. This direction is determined by the initial velocity, $v(0_+)$, and the types of scatterers that are located initially at $z = \pm 1$.

The set of initial configurations for which the first statement of the theorem fails consists of all the configurations with the positive or negative tail of forward scatterers. This set has a zero measure in the set of all initial configurations.

The second theorem describes the statistical properties of the NOS models.

Theorem 3.2.2. *If the value of the rigidity r is even, then for all t, $\mathbb{E}z(t) = 0$. Moreover, there exist $T = T(q, r) \in \mathbb{N}$, $C_1 = C_1(q) > 0$, and $C_2 = C_2(q) > 0$ such that*

$$C_1 \log t \leq \mathbb{E}z^2(t) \leq C_2 \log t, \quad \forall t \geq T.$$

Otherwise, if the value of the rigidity r is odd then there exist $C = C(q) > 0$ and $T = T(q, r) > 0$, such that

$$|\mathbb{E}z(t)| = C \frac{t}{r}, \quad \forall t \geq T.$$

Next, we write down the conditions, that determine the direction of propagation for the case of odd rigidity.

Lemma 3.2.3. *Suppose that rigidity r is odd. Then for sufficiently large $T = T(q, r)$ (as in the second part of the preceding theorem) we have the following:*

(1) $\mathbb{E}v(t) > 0$, $\forall t \geq T$ *if either*
- $v(0_+) = +1$ *and* $\omega(0, 0) = \text{'BS'}$ *or*

- $v(0_+) = +1$ *and* $\omega(0,0) = \omega(+1,0) = '\texttt{FS}'$ *or*
- $v(0_+) = -1$, $\omega(0,0) = '\texttt{FS}'$, $\omega(-1,0) = '\texttt{BS}'$;

(2) $\mathbb{E}v(t) < 0$, $\forall t \geq T$ *if either*

- $v(0_+) = +1$, $\omega(0,0) = '\texttt{FS}'$, $\omega(+1,0) = '\texttt{BS}'$ *or*
- $v(0_+) = -1$ *and* $\omega(0,0) = \omega(-1,0) = '\texttt{FS}'$ *or*
- $v(0_+) = -1$ *and* $\omega(0,0) = '\texttt{BS}'$.

4. Walks in Rigid Environments with aging on $\mathbb{Z}^1$

In this section we study models in environments with aging. Again, we consider separately OS- and NOS-models.

4.1. OS Model. We will show that the dynamics of this model is qualitatively similar to the case of constant rigidity. Assume that initial configuration of scatterers is such that there is no positive tail of right scatterers nor negative tail of left scatterers. The set of such configurations has a full measure.

First, we obtain some new results for the models with constant rigidity that are required for analysis of the models with aging, i.e. with varying rigidity.

Lemma 4.1.1. *Consider a walk in rigid environment with a constant rigidity on* $\mathbb{Z}^1$. *Then, there exist* $T_0 = 0 \leq T_1 \leq T_2 \leq \ldots$, $T_j = T_j(q, r_j) \in \mathbb{N}$ *such that* $\forall r_j \in \mathbb{N}$ *one can find* $C_1 = C_1(q) > 0$ *and* $C_2 = C_2(q) > 0$ *with the following property*

$$\mathbb{E}z_j(t) = 0, \quad C_1 \frac{t}{r_j} \leq \mathbb{E}z_j^2(t) \leq C_2 \frac{t}{r_j}, \quad \forall t \geq T_j, \quad \forall j \in \mathbb{N},$$

where z_j *is a position of a particle at the time* t *in a system with a constant rigidity* $r(z,t) = r_j$.

Proof. By Theorem 3.1.2 for each fixed $j \in \mathbb{N}$ one has $\mathbb{E}z_j(t) = 0$ for all values of $t \in \mathbb{N}$. Moreover, there exists $T_j = T_j(q, r_j)$ such that for all $t \geq T_j$ we can find two constants $C_1 = C_1(q) > 0$ and $C_2 = C_2(q) > 0$ (both independent of r_j) such that

$$C_1 \frac{t}{r_j} \leq \mathbb{E}z_j^2(t) \leq C_2 \frac{t}{r_j}, \quad \forall t \geq T_j.$$

We can apply this argument for all values of $j \in \mathbb{N}$ and obtain the sequence $\{T_j(q, r_j)\}_{j=0}^{\infty}$ with the same two constants $C_1 = C_1(q) > 0$ and $C_2 = C_2(q) > 0$ for all values of j. Thus the result follows. □

Finally, we consider a model with aging. Suppose that the rigidity function is defined as

$$r(z,t) = r(t) = r_j, \quad j \in \mathbb{Z}^+, \quad \forall z \in \mathbb{Z}, \quad t \in [\tau_{j-1}, \tau_j) \cap \mathbb{N}, \tag{4.1}$$

for some $\tau_0 = 0 \le \tau_1 \le \tau_2 \le \ldots$, $\tau_j \in \mathbb{N}$ to be specified below and $r_j \in \mathbb{N}$, $r_j \le r_{j+1}$.

The next theorem provides the results on the statistical properties of the OS models with aging.

Theorem 4.1.2. *Consider the OS model with the rigidity function defined by 4.1. Then, one has*

$$\mathbb{E}z(t) = 0, \quad \forall t > 0.$$

Moreover, for any $\tau_1 \le \tau_2 \le \ldots$, $\tau_j \in \mathbb{N}$, *with the property*

$$\tau_j \ge T_{j+1}, \quad j = 1, 2, \ldots \tag{4.2}$$

there exist $C_1 = C_1(1) > 0$ *and* $C_2 = C_2(q) > 0$ *such that* $\forall n \ge 1$,

$$C_1 \left[\sum_{j=1}^{n-1} \left(\frac{1}{r_j} - \frac{1}{r_{j+1}} \right) \tau_j + \frac{\tau_n}{r_n} \right] \le \mathbb{E}z^2(\tau_n) \le$$
$$C_2 \left[\sum_{j=1}^{n-1} \left(\frac{1}{r_j} - \frac{1}{r_{j+1}} \right) \tau_j + \frac{\tau_n}{r_n} \right].$$

Proof. Theorem 3.1.2 states that the particle will oscillate regardless of the value of the rigidity and that $\mathbb{E}z(t) = 0$, $\forall t > 0$.

Suppose that $\tau_j \ge T_{j+1}, \quad j = 1, 2, \ldots$ and let us now compute $\mathbb{E}z^2(\tau_n)$ for all n.

At $t = \tau_1$, by Theorem 3.1.2, there exist two constants $C_1 = C_1(1) > 0$ and $C_2 = C_2(q) > 0$ such that

$$C_1 \frac{\tau_1}{r_1} \le \mathbb{E}z^2(\tau_1) \le C_2 \frac{\tau_1}{r_1}.$$

At the moment of time $t = \tau_1$ we change the rigidity from r_1 to r_2. When the rigidity of the environment is $r(t) = r_1$ the distance a particle travels in τ_1 units of time equals to the distance the particle travels in $\frac{r_2}{r_1}\tau_1$ units of time but with the rigidity of the environment $r(t) = r_2$. This follows from the relations

$$C_1 \frac{\frac{r_2}{r_1}\tau_1}{r_2} = C_1 \frac{\tau_1}{r_1} \le \mathbb{E}z_1^2(\tau_1) \le C_2 \frac{\tau_1}{r_1} = C_2 \frac{\frac{r_2}{r_1}\tau_1}{r_2},$$

and

$$C_1 \frac{\frac{r_2}{r_1}\tau_1}{r_2} \le \mathbb{E}z_2^2 \left(\frac{r_2}{r_1}\tau_1 \right) \le C_2 \frac{\frac{r_2}{r_1}\tau_1}{r_2}.$$

Next, particle continues its movement on the lattice with a new rigidity, r_2. So the particle first travels τ_1 units of times when the rigidity of environment equals r_1 or, equivalently, for $\frac{r_2}{r_1}\tau_1$ units of time when the rigidity of environment equals r_2, and $\tau_2 - \tau_1$ units of time when the rigidity of environment equals r_2.

Hence, at the moment of time $t = \tau_2$ the following relationship holds

$$C_1 \frac{\frac{r_2}{r_1}\tau_1 + (\tau_2 - \tau_1)}{r_2} \leq \mathbb{E}z^2(\tau_2) \leq C_2 \frac{\frac{r_2}{r_1}\tau_1 + (\tau_2 - \tau_1)}{r_2}.$$

Therefore,

$$C_1 \left[\left(\frac{1}{r_1} - \frac{1}{r_2} \right) \tau_1 + \frac{\tau_2}{r_2} \right] \leq \mathbb{E}z^2(\tau_2) \leq C_2 \left[\left(\frac{1}{r_1} - \frac{1}{r_2} \right) \tau_1 + \frac{\tau_2}{r_2} \right].$$

We continue inductively to finish the proof. Suppose that at the time $t = \tau_n$ we have the following bounds

$$C_1 \left[\sum_{j=1}^{n-1} \left(\frac{1}{r_j} - \frac{1}{r_{j+1}} \right) \tau_j + \frac{\tau_n}{r_n} \right] \leq \mathbb{E}z^2(\tau_n) \leq$$
$$C_2 \left[\sum_{j=1}^{n-1} \left(\frac{1}{r_j} - \frac{1}{r_{j+1}} \right) \tau_j + \frac{\tau_n}{r_n} \right].$$

Then by the same argument as above one gets at time $t = \tau_{n+1}$ that

$$\frac{C_1}{r_{n+1}} \left(r_{n+1} \left[\sum_{j=1}^{n-1} \left(\frac{1}{r_j} - \frac{1}{r_{j+1}} \right) \tau_j + \frac{\tau_n}{r_n} \right] + \tau_{n+1} - \tau_n \right) \leq$$
$$\mathbb{E}z^2(\tau_n) \leq$$
$$\frac{C_2}{r_{n+1}} \left(r_{n+1} \left[\sum_{j=1}^{n-1} \left(\frac{1}{r_j} - \frac{1}{r_{j+1}} \right) \tau_j + \frac{\tau_n}{r_n} \right] + \tau_{n+1} - \tau_n \right).$$

or, equivalently,

$$C_1 \left[\sum_{j=1}^{n} \left(\frac{1}{r_j} - \frac{1}{r_{j+1}} \right) \tau_j + \frac{\tau_{n+1}}{r_{n+1}} \right] \leq \mathbb{E}z^2(\tau_{n+1}) \leq$$
$$C_2 \left[\sum_{j=1}^{n} \left(\frac{1}{r_j} - \frac{1}{r_{j+1}} \right) \tau_j + \frac{\tau_{n+1}}{r_{n+1}} \right].$$

□

Observe that $\mathbb{E}z^2(t)$ grows linearly in time, similarly to the case of constant rigidity.

4.2. **NOS Model.** We suppose again that the rigidity function has the form

$$r(z,t) = r(t) = r_j, \quad j \in \mathbb{Z}^+, \quad \forall z \in \mathbb{Z}, \quad t \in [\tau_{j-1}, \tau_j) \cap \mathbb{N}, \tag{4.3}$$

for some $\tau_0 = 0 \leq \tau_1 \leq \tau_2 \leq \ldots, \tau_j \in \mathbb{N}$ to be defined below and $r_j \in \mathbb{N}$, $r_j \leq r_{j+1}$.

The dynamics of NOS model with aging is more complicated than the one of the OS model with aging. We first consider the case when a parity of rigidity $r(t)$ does not change.

4.2.1. *Even rigidity.* We assume that r_j is even for all $j \in \mathbb{N}$. First, we look at the family of systems with the constant rigidity.

Lemma 4.2.1. *There exist $T_1, T_2, \ldots, T_j = T_j(q, r_j) \in \mathbb{N}$ such that for each $r_j \in \mathbb{N}$ one has $\mathbb{E}z_j = 0$. Moreover, one can find two constants $C_1 = C_1(q) > 0$ and $C_2 = C_2(q) > 0$ satisfying the relationship*

$$C_1 \log t \leq \mathbb{E}z_j^2(t) \leq C_2 \log t, \quad \forall t \geq T_j, \quad \forall j \geq 1,$$

where $z_j(t)$ is a position of a particle at the time t in a system with a constant rigidity $r(z,t) = r_j$.

Proof. By Theorem 3.2.2 for each fixed $j \in \mathbb{N}$ one has $\mathbb{E}z_j(t) = 0$ for all values of $t \in \mathbb{N}$. Moreover, there exists $T_j = T_j(q, r_j)$ such that for all $t \geq T_j$ we can find two constants $C_1 = C_1(q) > 0$ and $C_2 = C_2(q) > 0$ (both independent of r_j) such that

$$C_1 \log t \leq \mathbb{E}z_j^2(t) \leq C_2 \log t, \quad \forall t \geq T_j.$$

We can apply this result for all values of $j \in \mathbb{N}$ and pick the sequence $\{T_j(q, r_j)\}_{j=0}^{\infty}$ and the same two constants $C_1 = C_1(q) > 0$ and $C_2 = C_2(q) > 0$ for all values of j. Thus the result follows. □

Now we go back to the systems with aging.

Theorem 4.2.2 (NOS model, even rigidity)**.** *Suppose that r_j is even for all $j \in \mathbb{N}$ and $\tau_j \geq T_j$ for all $j \in \mathbb{N}$. Then there exist two numbers $C_1 = C_1(q) > 0$ and $C_2 = C_2(q) > 0$ such that*

$$C_1 \log \tau_n \leq \mathbb{E}z^2(\tau_n) \leq C_2 \log \tau_n, \quad \forall n \in \mathbb{N}.$$

Proof. The geometric structure of the orbits does not change if we keep the parity constant. Thus if all r_j's are even then almost all orbits will oscillate. Consider the collection of these orbits.

Suppose that

$$\tau_j \geq T_j, \quad \forall j \in \mathbb{N}.$$

Then

$$\mathbb{E}z_n^2(\tau_n) \leq \mathbb{E}z^2(\tau_n) \leq \mathbb{E}z_1^2(\tau_n),$$

where, as above, $z_j(t)$ is a position of the particle at the time t in a system with constant rigidity $r(z,t) = r_j$. Since $\tau_n \geq T_n$ by the preceding lemma there are $C_1 = C_1(q) > 0$ and $C_2 = C_2(q) > 0$ such that

$$C_1 \log \tau_n \leq \mathbb{E}z_n^2(\tau_n) \leq \mathbb{E}z^2(\tau_n) \leq \mathbb{E}z_1^2(\tau_n) \leq C_2 \log \tau_n.$$

Thus the result follows. □

Hence, if we allow enough time for a system to evolve (i.e. each $T_j(q, r_j)$ is large enough) then the behavior of the systems with aging is similar to the ones with constant rigidity.

4.2.2. *Odd rigidity.* We assume that r_j is odd for all $j \in \mathbb{N}$. Again, at first, we look at the family of systems with the constant rigidity.

Lemma 4.2.3. *There exist $T_1, T_2, \ldots, T_j = T_j(q, r_j) \in \mathbb{N}$ such that for each $r_j \in \mathbb{N}$, one can find two constants $C_1 = C_1(q) > 0$ and $C_2 = C_2(q) > 0$ satisfying relationship*

$$C_1 \frac{t}{r_j} \leq |\mathbb{E}z_j(t)| \leq C_2 \frac{t}{r_j}, \quad \forall t \geq T_j, \quad \forall j \geq 1,$$

where z_j is a position of the particle at the time t in a system with a constant rigidity $r(z,t) = r_j$.

Proof. By Theorem 3.2.2 for each fixed $j \in \mathbb{N}$ there exists $T_j = T_j(q, r_j)$ such that for all $t \geq T_j$ we can find two constants $C_1 = C_1(q) > 0$ and $C_2 = C_2(q) > 0$ (both independent of r_j) such that

$$C_1 \frac{t}{r_j} \leq |\mathbb{E}z_j(t)| \leq C_2 \frac{t}{r_j}, \quad \forall t \geq T_j.$$

We can apply this result for all values of $j \in \mathbb{N}$ and obtain the sequence $\{T_j(q, r_j)\}_{j=0}^{\infty}$ and the same two constants $C_1 = C_1(q) > 0$ and $C_2 = C_2(q) > 0$ for all values of j. Thus the result follows. □

Next, we switch to the systems with aging.

Theorem 4.2.4 (NOS model, odd rigidity)**.** *Suppose that r_j is odd for all $j \in \mathbb{N}$ and $\tau_j - \tau_{j-1} \geq T_j$ for all $j \in \mathbb{N}$. Then there are $C_1 = C_1(q) > 0$ and $C_2 = C_2(q) > 0$ such that*

$$C_1 \sum_{j=1}^{n} \frac{\tau_j - \tau_{j-1}}{r_j} \leq |\mathbb{E}z(\tau_n)| \leq C_2 \sum_{j=1}^{n} \frac{\tau_j - \tau_{j-1}}{r_j}, \quad \forall n \in \mathbb{N}.$$

Proof. During each interval $[\tau_{j-1}, \tau_j)$ of constant rigidity $r(z,t) = r_j$, $t \in [\tau_{j-1}, \tau_j)$, by Lemma 4.2.3 we have the following estimate

$$C_1 \frac{\tau_j - \tau_{j-1}}{r_j} \leq |\mathbb{E}z(\tau_j) - \mathbb{E}z(\tau_{j-1})| \leq C_2 \frac{\tau_j - \tau_{j-1}}{r_j}.$$

Since particle propagate into one direction on each of these intervals the result follows. □

Again, we have a behavior similar to the systems with constant rigidity: if we allow enough time for each interval of constant rigidity then we observe a propagation with $\mathbb{E}z(\tau_n)$ growing linearly in time.

4.2.3. *Alternating parity of the rigidity.* Assume now that r_{2j} is even and r_{2j+1} is odd for all $j \in \mathbb{N}$. Also, assume that $\tau_1 = 0$, that is we start with even rigidity (which corresponds to an oscillatory behavior). We will alter now the way rigidity changes, namely, we now assume that at the time τ_j (the moment of time we change a value of rigidity) we reset the index of all sites to zero, i.e. $\eta(z, \tau_{j+}) = 0$, $\forall z \in \mathbb{Z}$, $\forall j \in \mathbb{N}$.

Assume that τ_j satisfies the inequality

$$\tau_j - \tau_{j-1} \geq T_j, \quad j \geq 2, \quad \tau_1 = 0, \tag{H1}$$

where T_j defined as in Lemmas 4.2.1 or 4.2.3 (depending on parity of j). That means that on each interval of time $[\tau_j, \tau_{j+1})$ on which rigidity is constant, we can apply the asymptotic estimates obtained earlier.

Also, assume that

$$\tau_{2j+1} - \tau_{2j} \gg \log\left(\tau_{2i} - \tau_{2i-1}\right), \quad \forall i, j \geq 1, \quad \tau_1 = 0. \tag{H2}$$

This relationship says that the time intervals when a particle oscillates are much shorter compare to the time intervals when the particle propagates in one direction. This ensures that two consecutive oscillations do not overlap.

First, we consider the interfaces between intervals of time with even rigidity and with odd rigidity.

Lemma 4.2.5. *Suppose that at the time $t = \tau_{2n}$ rigidity switches from the even value $r = 2n$ to the odd value $r = 2n+1$. Then the probability that $v(\tau_{2n+}) = +1$ is the same as the probability that $v(\tau_{2n+}) = -1$ and equals to $\frac{1}{2}$.*

Proof. During the times when rigidity is even particle undergoes oscillations with a zero mean. Thus the average velocity is zero. The result follows. □

We now know that when rigidity switches to the an value, the particle will have the same probability of moving in either direction. During the time interval when rigidity is an odd number the particle eventually propagate in one direction. Combing the results from Lemma 3.2.3 and Lemma 4.2.5 we obtain the following statement.

Lemma 4.2.6. *There exists increasing sequence of natural numbers* $\{\hat{T}_{2j}\}_{j=1}^{\infty}$ *such that if*

$$\tau_{2j} - \tau_{2j-1} > \hat{T}_{2j}, \quad \forall j \geq 1, \tag{H3}$$

then the probability of

$$\mathbb{E}\left[z(t_{2j+1}) - z(t_{2j})\right] > 0$$

is the same as probability of

$$\mathbb{E}\left[z(t_{2j+1}) - z(t_{2j})\right] < 0$$

and is equal to 0.5*.*

Hence, if we pick the sequence $\{\tau_j\}$ in such a way so it satisfies conditions H1 - H3, when the value of the rigidity switches from even to odd number with the probability 0.5 the particle will propagate into one or another direction. Thus we have a situation resembling a standard random walk.

The oscillatory behavior during the intervals of time when rigidity is even plays a role of a "dice" (i.e. it randomizes the motion of the particle), but unlike the standard random walk the decision about the direction of the eventual propagation during the stage with odd rigidity is not made instantaneously. Instead, it takes $\tau_{2j}-\tau_{2j-1}$ units of time to make a decision. By Lemma 4.2.3 the displacement during the intervals of odd rigidity equals

$$|\mathbb{E}z(\tau_{2n+1}) - \mathbb{E}z(\tau_{2n})| \sim \frac{\tau_{2n+1} - \tau_{2n}}{r_{2n+1}}. \tag{4.4}$$

5. Concluding Remarks

We introduced a broad class of models which describe propagation of signals (particles, waves, information, etc) in discrete environments with aging. An environment could be any graph $\mathcal{G}$ (directed or undirected). The process of aging is described by a time-dependent rigidity of an environment. The exact results obtained for the case when $\mathcal{G}$ is $\mathbb{Z}^1$ and the rigidity is a piecewise constant function of time. It is unlikely that sufficiently complete analytical results could be obtained for the general class of graphs (general structures of networks). However, these models are very amendable for numerical studies. Therefore, it seems feasible that deterministic walks in random environments with aging could provide a lot of useful information about the behavior of systems with aging.

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References

[1] B. Abou, F. Gallet, Probing an nonequilibrium Einstein relation in an aging colloidal glass, Phys. Rev. Lett., 93 (2004), 160603.

[2] L. Bellon, S. Ciliberto, Experimental study of fluctuation dissipation relation during the aging process, Physica D, 168 (2002), 325-333.

[3] L. Bellon, S. Ciliberto, C. Laroche, Advanced Memory effects in the aging of polymer glass, Eur. Phys. J. B., 25 (2002), 223-228.

[4] J. P. Bouchaud, Weak ergodicity breaking and aging in disordered systems, J. Phys. I France, 2 (1992), 1705-1714.

[5] L. Buisson, L. Bellon, S. Ciliberto, Intermittency in aging, J. Physics: Condens. Matter, 15 (2003), s1163-s1179.

[6] L. A. Bunimovich, Walks in rigid environments, Phys. A 279 (2000), no. 1-4, 169-179.

[7] L. A. Bunimovich, Walks in rigid environments: symmetry and dynamics, Astérisque No. 286 (2003), 231-248.

[8] L. A. Bunimovich, Deterministic walks in random environments, Phys. D 187 (2004), no. 1-4, 20-29.

[9] L. A. Bunimovich and A. Yurchenko, Deterministic Walks in Markov environments with constant rigidity (2005, AMS Contemporary Mathematics Series, accepted).

[10] I. D. Cousin, J. Krause, R. James, G. D. Ruxton, N. R. Franks, Collective memory and spatial sorting in animal groups, J. Theor. Biology, 218 (2002), 1-11.

[11] A. Crisanti, F. Ritort, Intermittency of Glassy relaxation and the emergence of nonequilibrium spontaneous measure in the aging regime, Europhys. Lett, 66 (2004), 253-258.

[12] D. Herrisson, M. Ocio, Fluctuation-dissipation ratio of a spin glass in the aging regime, Phys. Rev. Lett., 88 (2002), 257702.

[13] H. Nyquist, Thermal agitation of electrical charge in conductors, Phys Rev., 32 (1928), 110-113.

[14] L. C. Struick, Physical aging in amorphous polymers and other materials, Elsevier, Amsterdam, 1978.

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